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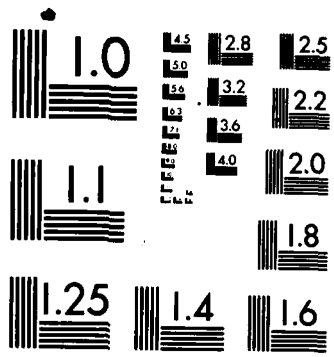

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ATLANTA, GEORGIA 30332

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NEW CONFIDENCE INTERVAL ESTIMATORS USING STANDARDIZED TIME SERIES

by

David Goldsman

and

Lee Schruben

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Abstract

We develop new confidence interval estimators for the underlying mean of a stationary simulation process. These estimators can be viewed as generalizations of Schruben's so-called *standardized time series area* confidence interval estimators. Various properties of the new estimators are given.

In this note, we present new confidence interval estimators for the underlying mean μ of a stationary simulation process. These estimators can be viewed as generalizations of the so-called *area* estimators given in Schruben (1983) and Goldsman (1984).

Consider the stochastic process $X_1,\ldots,X_m$. Define

$$\bar{X}_j \equiv \frac{1}{j}\sum_{k=1}^{j} X_k \ , \quad j=1,\ldots,m,$$

$$\sigma^2 \equiv \lim_{m\to\infty} m\mathrm{Var}(\bar{X}_m) \ , \quad \text{and}$$

$$T_m(t) \equiv \frac{\lfloor mt\rfloor(\bar{X}_m-\bar{X}_{\lfloor mt\rfloor})}{\sigma\sqrt{m}} \ , \quad 0\le t\le 1,$$

where $\lfloor\cdot\rfloor$ is the greatest integer function. $\{T_m(t),\ 0\le t\le 1\}$ is called the *standardized time series*. Schruben (1983) proves that if $X_1,\ldots,X_m$ is a stationary, φ-mixing, finite variance sequence of random variables (satisfying one other technical condition), then as $m\to\infty$, $T_m(t)$ converges in distribution to a *standard Brownian bridge* process, $\{B_t,\ 0\le t\le 1\}$. Also, the standardized time series is asymptotically independent of $m\bar{X}_m$.

<u>Remark</u>: It is well known that $B_t \sim \mathrm{Nor}(0,t(1-t))$ and $\mathrm{Cov}(B_{t_1},B_{t_2}) = \min(t_1,t_2)(1-\max(t_1,t_2))$.

Define $A \equiv \frac{\sigma}{m}\sum_{k=1}^{m} c_k T_m(k/m)$, the c_k's being pre-specified. For large m, $A \approx$ (is approximately distributed as) $\sigma\mathrm{Nor}(0,V)$, where

$$V \equiv \frac{1}{m^2}\sum_{j=1}^{m}\sum_{k=1}^{m} \mathrm{Cov}(c_j B_{j/m}, c_k B_{k/m}) \ .$$

It will be computationally convenient to approximate V by letting $j = ms$ and $k = mt$ (so that $djdk = m^2 dsdt$). Thus,

$$V = \frac{1}{m^2} \sum_{j=1}^{m} \sum_{k=1}^{m} c_j c_k \min(j/m,k/m)[1-\max(j/m,k/m)]$$

$$\doteq \int_0^1 \int_0^1 c_{ms} c_{mt} \min(s,t)[1-\max(s,t)]\, dsdt$$

$$= 2 \int_0^1 \int_0^t c_{ms} c_{mt}\, s(1-t)\, dsdt \,. \tag{1}$$

So $A^2/V \doteq \sigma^2 \chi^2(1)$, and this is called the *weighted area estimator* for the variance σ^2.

Suppose now that we work with the process $X_1,\ldots,X_n$, where $n=bm$, and that this series satisfies Schruben's conditions. Divide the process into b contiguous *batches*, each of size m; i.e., $X_{(i-1)m+1}$, $X_{(i-1)m+2}$, ..., X_{im} comprise batch i, $i=1,\ldots,b$. Each individual batch can be standardized: For $i=1,\ldots,b$ and $j=1,\ldots,m$, let

$$\bar{X}_{i,j} \equiv \frac{1}{j} \sum_{k=1}^{j} X_{(i-1)m+k} \qquad \text{(average of the first } j\ X\text{'s from the } i\text{-th batch)},$$

$$\bar{\bar{X}}_n \equiv \frac{1}{n} \sum_{k=1}^{n} X_k \quad \text{(grand mean)},$$

$$T_{i,m}(t) \equiv \frac{\lfloor mt \rfloor (\bar{X}_{i,m} - \bar{X}_{i,\lfloor mt \rfloor})}{\sigma\sqrt{m}}\,, \quad 0 \le t \le 1, \quad \text{and}$$

$$A_i \equiv \frac{\sigma}{m} \sum_{k=1}^{m} c_k T_{i,m}(k/m) \,.$$

For large enough m, each of the standardized time series [the $T_{i,m}(t)$'s] is approximately distributed as a Brownian bridge; so $A_i \doteq \sigma \mathrm{Nor}(0,V)$, $i=1,\ldots,b$. Further, for large m, we can treat the batches as if they were (approximately) independent. This yields:

$$\frac{1}{V}\sum_{i=1}^{b} A_i^2 \;\Rightarrow\; \sigma^2\chi^2(b) .$$

An immediate consequence of Theorem 21.1 of Billingsley (1968) is that

$$Z_n \equiv \frac{\bar{\bar{X}}_n-\mu}{\sigma/\sqrt{n}} \;\Rightarrow\; \mathrm{Nor}(0,1) .$$

Then the asymptotic independence result above gives:

$$\frac{Z_n}{\left[\dfrac{\Sigma A_i^2}{\sigma^2 bV}\right]^{1/2}} \;\Rightarrow\; \frac{\mathrm{Nor}(0,1)}{\left[\dfrac{\chi^2(b)}{b}\right]^{1/2}} \sim t(b) .$$

We finally obtain confidence interval estimators for μ:

$$\Pr\left\{ \mu \in \bar{\bar{X}}_n \pm t_{b,1-\alpha/2}\left[\frac{1}{nbV} \sum_{i=1}^{b} A_i^2 \right]^{1/2} \right\} \;\Rightarrow\; 1-\alpha , \qquad (2)$$

where $t_{b,\nu}$ is the upper-ν quantile of the $t(b)$ distribution.

We consider various choices for the weights (the c_k's). These choices and their resulting V's [from the integral approximation (1)] are summarized in Table 1.

Remarks:

(i) After choosing the weighting sequence $\{c_j\}$, the associated V from Table 1 is used in (2) to form a confidence interval estimator.

(ii) The calculation of V from (1) is straightforward (but sometimes tedious).

Example: For choice 3 (from Table 1),

$$V = 2\int_0^1\int_0^1 \frac{1}{s(1-s)}\,\frac{1}{t(1-t)}\, s(1-t)\, ds\, dt$$

$$= 2\int_0^1\int_0^t \frac{1}{(1-s)t}\, ds\, dt \;=\; -2\int_0^1 \frac{\ln(1-t)}{t}\, dt \;=\; \frac{\pi^2}{3}\,.$$

(iii) The variance estimator which results from choice 1 (the equal weighting case) is asymptotically the same (as $m\to\infty$) as the so-called *area* variance estimator from Schruben (1983).

(iv) For each standardized time series, $\{T_{i,m}(m)\}$, choice 2 grants greater weight for 'small' values of t. Choices 3 through 6 give comparatively little weight to the middle ($t \approx 1/2$) of each standardized time series.

(v) Table 2 summarizes an empirical study involving the order 1 exponential autoregressive model [cf. Lewis, (1980)]. The weighted area estimators are seen to perform well for 'large' batch size.

(vi) Denote the random variable corresponding to the half-length of the weighted area estimator by H. Following Schmeiser (1982) and Goldsman and Schruben (1984) (G-S), it is easy to derive the following:

$$E[H] = \frac{\sigma}{\sqrt{n}}\, t_{b,1-\alpha/2}(2/b)^{1/2}\,\frac{\Gamma((b+1)/2)}{\Gamma(b/2)}\,,$$

$$Var(H) = \frac{\sigma^2}{n}\, t^2_{b,1-\alpha/2}\left\{1 - \frac{2}{b}\left[\frac{\Gamma((b+1)/2)}{\Gamma(b/2)}\right]^2\right\}, \text{ and}$$

The coverage probability,

$$Pr\{|\bar{\bar{X}}_n-\mu_1| < H\} = F(t_{b,1-\alpha/2}) - F(-t_{b,1-\alpha/2})\ , \text{ where}$$

$\Gamma(.)$ is the gamma function and F(.) is the c.d.f. of the noncentral

of correlation amongst batches which are encountered when using the area estimator.

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Table 1 - Choices of Weights and Resulting V's

weighting choice	c_j, $j=1,\ldots,m$	V from (1)
1	1	$1/12$
2	$1-\frac{j}{m}$	$1/45$
3	$\left[\frac{j}{m}\left(1-\frac{j}{m}\right)\right]^{-1}$	$\frac{\pi^2}{3}$
4	$\left\lvert\frac{1}{2}-\frac{j}{m}\right\rvert+\epsilon \quad (\epsilon\geq 0)$	$\frac{1}{320}+\frac{\epsilon}{32}+\frac{\epsilon}{12}$
5	$\left[\frac{1}{2}-\frac{j}{m}\right]^2+\epsilon \quad (\epsilon\geq 0)$	$\frac{1}{4032}+\frac{\epsilon}{120}+\frac{\epsilon^2}{12}$

Table 2

Performance of weighted area confidence interval estimators for the mean of an EAR(1) process with coefficient ρ = 0.2 and exponential (mean=1) noise based on 100 independent runs of 2560 observations each. [Choices of weights are summarized in Table 1].

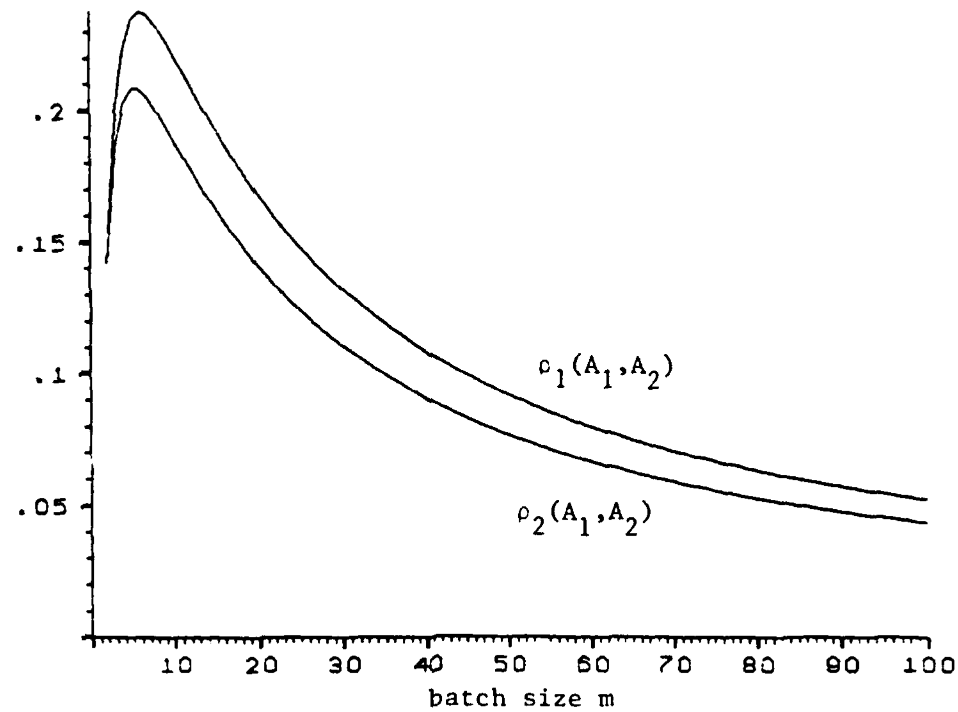
b	m \ weighting choice	1	2	3	4(ϵ=0)	5(ϵ=0)
	Confidence interval achieved coverage (90% desired)					
1	2560	.88	.92	.90	.89	.90
2	1280	.93	.96	.94	.93	.95
5	512	.95	.94	.96	.95	.96
10	256	.93	.93	.94	.95	.96
20	128	.93	.94	.91	.91	.91
	Average confidence interval half-length (x 10000)					
1	2560	1193	1266	1184	1199	1202
2	1280	592	612	598	613	617
5	512	480	481	486	492	498
10	256	440	441	436	444	446
20	128	426	423	415	427	426
	Sample standard deviation of half-lengths (x 10000)					
1	2560	900	923	876	932	931
2	1280	297	300	298	294	298
5	512	153	149	150	150	149
10	256	103	104	97.5	95.6	93.7
20	128	52.6	50.7	57.9	59.5	65.5

Table 3 - Typical Small-Sample Values of ρ_1 and ρ_2

(m = batch size)

α	m	ρ_1	ρ_2
0.5	15	-0.04268	-0.03686
0.5	20	-0.03231	-0.02767
0.5	25	-0.02600	-0.02215
0.0	--	0	0
-0.5	15	0.19445	0.16427
-0.5	20	0.16964	0.14261
-0.5	25	0.14969	0.12550
-0.9	15	0.42568	0.35214
-0.9	25	0.44111	0.36053
-0.9	50	0.43968	0.35646
-0.9	100	0.41352	0.33456

Figure 1: Correlation of A_1, A_2 versus batch size m for an MA(1) process with $\alpha = -0.5$



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